Lecture 4 ***Investigation of nonlinear ACS in phase plane***

The phase plane method was developed by Russian scientist *A. A. Andronov (А.А. Андронов).*

This method is intended for research of linear and nonlinear ACS of the 2nd order, but it is not a restriction, because it is always possible to decrease the order of mathematical equations for system description.

What is phase space? Let us consider a system of the following type:

$$T\,\dot{x} + x\,(t) = u\,(t).$$

The behavior of this system corresponds to the solution of homogeneous differential equation:

$$T\,\dot{x} + x\,(t) = 0\,.$$

In phase space two phase variables are used: the basic (fundamental) coordinate *x(t)* and its speed change $\dot{x}(t)$. In this case it will be a straight line with the angle of declination $\alpha = 1 / T$ (fig.6.10).

In this case coordinates $\dot{x}(t)$, *x(t)* are called *phase coordinates*. A *phase trajectory* is a line, which shows transition of figured point on the phase plane. Any point on the phase trajectory is called *figured point M*. The crossing of the phase trajectory with initial coordinates is called *special point* or *rest point*.

Assemblage of phase trajectories is called *phase portrait* of the system. It characterizes properties of the system. Here are the distinctive features of phase trajectories:

– phase trajectories do not cross each other anywhere expect at *special point, rest points.*

– phase trajectories can be built qualitatively;

– direction of phase trajectories must be shown by arrows;

– phase trajectories occupy all phase space.

In phase space time parameter is excluded. Indirectly it is reflected in a following way: to every moment t_k there corresponds a fixed value of coordinates $x(t_k)$ and $\dot{x}(t)$, shown by a point on the phase trajectory (fig.6.10)

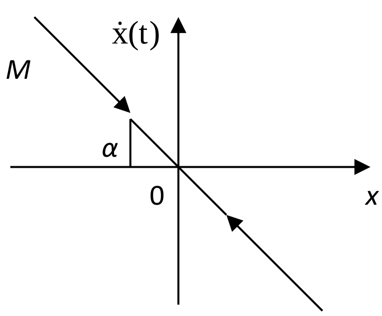

Fig.6.10. Example of a phase trajectory

Let's write the 2nd order equation in a scalar form:

$$\begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases}.$$

Then we divide the second equation by the first one and get *differential equation of phase trajectory:*

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)} \text{ or } \frac{dx_2}{dx_1} = F(x_1, x_2). \qquad (6.3)$$

The solution of the differential equation of the phase trajectories has the following form:

$$x_2 = f(x_1). \qquad (6.4)$$

Equation (6.4) is called an *equation of phase trajectory*.

Example 6.1. Let a system of the following form is given:

$$T_1 \ddot{x} + T_2 \dot{x} + x = 0, \qquad (*)$$

where $T_1 > 0$, $i = \overline{1,2}$; T_i are given constants;

It is necessary to get an equation of phase trajectory:

$$\frac{dx_2}{dx_1} = F(x_1, x_2); \quad x_2 = f(x_1)$$

at restriction $T_i \leq T_{i\ admit.}$ $\forall i = \overline{1,n}$.

Algorithm and solution

1. We introduce the following notations:

$$\begin{cases} x = x_1 \\ \dot{x} = x_2 \end{cases}.$$

2. Then equation (*) will be rewritten as the following:

$$T_1 \dot{x}_2 + T_2 x_2 + x_1 = 0.$$

3. After dividing $\frac{dx_1}{dt} = x_2 \neq 0$ we get:

$$T_1 \frac{\mathrm{dx}_2}{\mathrm{dt}} \frac{dt}{dx_1} + T_2 \frac{x_2}{x_2} + \frac{x_1}{x_2} = 0$$

$$T_1 \frac{\mathrm{d}x_2}{\mathrm{d}x_1} + T_2 + \frac{x_1}{x_2} = 0$$

4. Then the differential equation of phase trajectory will be:

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \left(\frac{x_1}{T_1 x_2} + \frac{T_2}{T_1} \right).$$

5. After integrating the last equation, we get a phase trajectory equation:

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = F(x_1, x_2); \int dx_2 = \int F(x_1, x_2) dx_1; x_2 = f(x_1).$$

A set of phase trajectories or a range of phase trajectories form a phase portrait. Each nonlinear ACS has its own unique portrait.

So, the first necessary characteristic of any ACS is its stability.

In a nonlinear system depending on the form of the differential equation of the phase trajectory $F(x_1, x_2)$ there exists one or many solutions, some of them can be stable, but the other are unstable. That is why in general case we cannot speak about stability or instability of a nonlinear system, we can speak only about stability or instability of its concrete motion or balanced condition.

About stability of nonlinear ACS we can judge on stability characteristics of linearized system in a close position near the balanced condition or near a rest point.

That is why further we will consider phase portrait of a linearized (linear) ACS. The condition showing possibility of dividing “on portraits” is the position of characteristic number in system.

Let’s consider phase portraits of a linear dynamical system of the 2nd order.